The Generator Paradox And Insights Into Faraday's Law

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[Adapted from a paper submitted to *The Physics Teacher* in 2001 based on calculations done many years earlier.]

How wonderful that we have met with a paradox. Now we have some hope of making progress.

—Niels Bohr[1]

Summary

I explain the surprising asymmetry in electromagnetic induction whereby an 'unconventional dynamo', in which the magnet rotates with respect to a stationary circuit, is not equivalent to a 'conventional dynamo', in which the circuit rotates with respect to a stationary magnet. I also give an example of an unusual asymmetric voltage measurement in which the recorded voltage depends on which side of the circuit the voltmeter is placed.

The Unconventional Dynamo

In the introductory paragraph of his epoch-making paper on the special theory of relativity[2], Einstein took note of what to him was a most important symmetry in the reciprocal action of a magnet and a conductor. 'The observable phenomenon,' Einstein wrote, 'depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion.' Einstein then went on to give the example that, if a magnet is in motion and a conductor at rest, an electric field is produced in the neighbourhood of the magnet that leads to a current in the conductor, whereas, if the magnet is stationary and the conductor is moving, an electromotive force (emf) is produced in the conductor giving rise to 'electric currents of the same path and intensity' as in the first case. From such considerations Einstein drew the conclusion that the phenomena of electrodynamics and mechanics 'possess no properties corresponding to the idea of absolute rest'.

An 'unconventional dynamo'[3] is a remarkably simple device that, to first appearances, violates the cardinal symmetry upon which special relativity is based. Were this actually the case, this modest bit of electrical circuitry would usher in a revolution in physics. This is, of course, not the case.

The essential features of the dynamo are illustrated in Figure 1. At the center is a cylindrical magnet whose outer surface is a north pole, such as would ideally result if an infinite number of one-dimensional magnetized needles were inserted south-pole first into a cylindrical cushion, each needle perpendicular to the cylinder axis. (The actual magnet of the cited reference was made by gluing rectangular Magnadur™ magnets to a wooden cylinder, each magnet with its north pole facing outward.)

[1] Quoted in *Niels Bohr : The Man, His Science, & the World They Changed* by Ruth Moore, (Knopf, 1966) p. 196.

[2] A. Einstein, "On the Electrodynamics of Moving Bodies", *Annalen der Physik* **17** (1905); reprinted in *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity,* Eds. W. Perrett and G. B. Jeffery (Dover Publications, New York, 1952) 37.

[3] W. Dindorf, "Unconventional Dynamo", *The Physics Teacher* **40** (April 2002) 220.

The magnet can be rotated about the vertical symmetry axis (the z-axis). Rectangle ACDJ represents a loop that can be rotated independently about nonconducting segment AJ on the z-axis. The segments AC, CD, and DJ drawn in heavy lines are conducting. The unlettered rectangle on the left-hand side of the figure, containing the galvanometer G, represents a fixed—i.e. nonrotatable—loop, the conducting portions of which are again shown as heavy lines. Together, the fixed and rotatable loops constitute a closed conducting path.

As reported, but not explained, in the cited reference, the 'unconventional dynamo' exhibited the following characteristics:

(A) If the magnet is rotated and the loop is stationary, no emf is registered on the galvanometer.
(B) If the magnet is stationary and the loop is rotated, the galvanometer registers an emf.
(C) If both the magnet and the loop are rotated together, the galvanometer registers an emf.

What, then, is one to make of a dynamo in which it would appear (contrary to Einstein) that the absolute motion of the loop, rather than the relative motion of the loop and magnet, leads to an observable phenomenon?

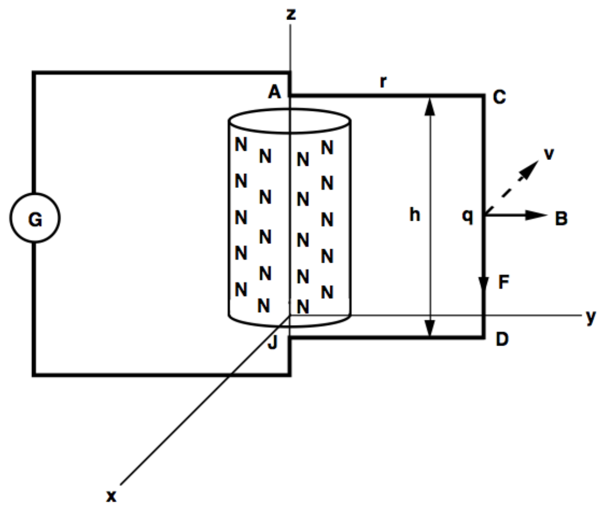

Figure 1: Unconventional dynamo consisting of a cylindrical magnet (with outward radially directed north pole) rotatable within the conducting loop marked with heavy lines. Alternatively, the rectangular area defined by the boundary ACDJ can be rotated about the cylinder axis (z-axis). Induced emf is registered by galvanometer G. A positive charge q moving with velocity **v** in the presence of magnetic field **B** experiences a Lorentz force **F** as shown.

There is, in fact, no violation of relativity because the actions represented in statements (A) and (B) do not constitute relative motions. In the one case (A), the magnet is rotating and the entire conducting loop is stationary. In the other case (B), the magnet is stationary, but only a portion of the conducting loop (segments AC, CD, DJ) is rotating. This asymmetry has experimental consequences which can be worked out readily by proper use of Faraday's law of induction.

Interlude on Faraday's Law

Phenomenologically, Faraday's law relates the time rate of change in magnetic flux Φ to the induced emf $\mathcal{E}$

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{1}$$

in which the flux through an *open* surface $\boldsymbol{\Sigma}$ is

$$\Phi = \iint_{\Sigma} \mathbf{B} \cdot \mathbf{n}\, dS \,. \tag{2}$$

In electrodynamics, as well as other branches of physics, the evaluation of integrals over areas depends critically on whether the surface is open or closed. An open surface (like a parabolic surface) does not have an inside or outside although one arbitrarily defines the 'outward' unit normal vector. A closed surface (like a spherical surface) has an inside and outside. Water contained in the inside would not spill out, no matter how the surface is oriented—a property not possessed by the parabolic surface.

Faraday's law of induction is a rather subtle law, requiring that one pay careful attention to certain geometric choices. The choices may be arbitrary, but must be used consistently. In Eq. (2) the integrand is the scalar product of the magnetic field vector **B** (technically, the magnetic induction) with a unit vector **n** defined to be the outward normal to the surface penetrated by the magnetic field lines. The selection of the outward normal direction then fixes the positive sense of circumnavigation of the boundary $\mathcal{C}$ of $\boldsymbol{\Sigma}$ by means of a right-hand rule (of which there are many in physics). In this case one puts the thumb of the right hand in the direction of **n**; the fingers curl in the positive sense of $\mathcal{C}$. Physically, this means that the induced emf would, according to Eq. (1), drive a current in the positive sense around the loop if (a) the flux through the loop were diminishing in time $\left(d\Phi/dt < 0\right)$ and (b) the boundary were actually conducting (as is the case for the dynamo of Figure 1). By "current" is here meant the standard convention of a flow of positively charged particles, rather than of negatively charged electrons, which is actually the case for a wire conductor.

Although Eq. (2) may look simple enough at first glance, there is more than meets the eye if the boundary or any part of the boundary of Σ is in motion. In that case reduction of the surface integral, taking account of the fact that the net magnetic flux through a *closed* surface must vanish (a statement expressing the absence of magnetic monopoles in nature, $\nabla \cdot \mathbf{B} = 0$), leads to Faraday's law in the form

$$\mathcal{E} = -\iint_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS + \oint_{\mathcal{C}} \mathbf{v} \times \mathbf{B} \cdot \mathbf{ds}\,. \tag{3}$$

The first integral on the right hand side is the contribution to the emf resulting from the variation (if any) of the magnetic field in time. The second integral, which is evaluated over the closed contour of Σ, is the contribution to the emf from flux through the moving boundary. The integrand contains the cross product of the velocity **v** of each path segment **ds** with the magnetic field at the location of **ds**. The scalar product shows that only the component of $\mathbf{v} \times \mathbf{B}$ tangential to the contour contributes to the emf.

The derivation of Eq. (3) is not difficult, but involves mathematical considerations that go beyond the intended scope of this paper and will be left to a reference.[4] The second integral on the right, however, is amenable to an interpretation in terms of the Lorentz magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. Figure 1 shows a positive point charge q at a location on the segment CD where the magnetic field is radially outward from the cylinder and the velocity of the charge, acquired by rotation of segment CD around the z-axis, is perpendicular to the plane of the loop and directed (at the instant shown) into the page. The Lorentz force accelerates the particle down the segment (i.e. in the direction from C to D), thereby raising the potential of point D relative to point C. By definition, the emf resulting from a force **F** is the work per charge $\oint_{\mathcal{C}} (\mathbf{F}/q) \cdot \mathbf{ds}$ to move a charge around the conducting loop, or in this case $\oint_{\mathcal{C}} \mathbf{v} \times \mathbf{B} \cdot \mathbf{ds}$, which is precisely the second integral in Eq. (3).

Richard Feynman, whose insightful *Lectures on Physics* have instructed many a graduate student and teacher over the years (although originally intended for undergraduates) has singled out the phenomenon of induction as conceptually unique. 'We know of no other place in physics,' Feynman wrote[5],

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[4]P. Lorrain and D. Corson, *Electromagnetic Fields and Waves* Second Edition (W. H. Freeman, San Francisco, 1970) 340.

[5]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics,* (Addison-Wesley, Reading, 1964) Vol. II, page 17-2.

> 'where such a simple and accurate general principal requires for its real understanding an analysis in terms of *two different phenomena*. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless in this case there does not appear to be any such profound implication. We have to understand the "rule" as the combined effects of two quite separate phenomena.' [Italics are Feynman's.]

It is somewhat difficult to know what to make of Feynman's declaration, for, as the ingenious originator of his own form of electrodynamics (employing waves that propagate both forward and backward in time) and as one of the principal contributors to the creation of quantum electrodynamics, he, of all people, must have been thoroughly conversant with a phenomenon as basic as induction. And yet, if by 'two quite separate phenomena', Feynman is saying that one must invoke the Lorentz force law to obtain the $\mathbf{v}\times\mathbf{B}$ term in the emf of a moving conductor, that is simply not the case.[6] The term arises naturally, as I have said, by taking account in Eqs (1) and (2) of the change in flux through the area generated by the moving boundary and by invoking a seminal property (the vanishing divergence) of magnetic fields.

Moreover, there is indeed a 'deep underlying principle' that relates Faraday's law in its integral form [Eq. (1)] and the Lorentz force law. That principle is the principle of special relativity.[7] The emf $\mathcal{E}$ in the left side of Eq. (3) is, as stated previously, an integral of the net force per charge $\mathbf{F}/q$ over the boundary through which the magnetic flux is changing. However, to be precise, $\mathbf{F}/q$ is the electric field $\mathbf{E}'$ as measured by an observer at rest with respect to the conductor. If the second term on the ride side of Eq. (3) is moved to the left side and the two integrals combined, the resulting integrand $\mathbf{E}'-\mathbf{v}\times\mathbf{B}$ is interpretable by relativity theory as the electric field **E** measured by an observer at rest with respect to the magnet (laboratory frame). The electric fields in the two frames are then related by

$$\mathbf{E}'=\mathbf{E}+\mathbf{v}\times\mathbf{B} \tag{4}$$

which is accurate to order v/c, with c the speed of light. The relation shown in Eq. (4) is not exact because the relativistic transformation of time between clocks at rest in the laboratory and clocks at rest relative to the conductor has been neglected. If there is no electric field in the laboratory frame (**E** = 0), as is the case of the unconventional dynamo under consideration, then the electric force $q\mathbf{E}'$ experienced by a charge in the rest frame of the conductor is interpreted as a magnetic (Lorentz) force $q\mathbf{v}\times\mathbf{B}$ from the perspective of a laboratory observer.

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[6]Interestingly, Feynman has erred in the interpretation of another fundamental phenomenon involving magnetic flux, the quantum mechanical phenomenon known as the Aharonov-Bohm (AB) effect. In the AB effect the magnetic flux itself—rather than the variation in time of the magnetic flux—can affect the location of fringes (i.e. maxima and minima) in an electron interference experiment. Feynman interpreted the effect in terms of the Lorentz force law (Ref. 6, pp. 15-12 to 15-14), but this is not correct. See M. P. Silverman (a) *More Than One Mystery: Explorations in Quantum Interference* (Springer-Verlag, New York, 1995) Chapter 1, or (b) *Quantum Superposition: Counterintuitive Consequences of Coherence, Entanglement, and Interference* (Springer, Heidelberg, 2008) Chapter 1.

[7]The situation is actually a little more complicated, for one must also invoke the Lorentz invariance of electric charge, a principle, seemingly independent of any other, stating that the electric charge of a particle is unaffected by its velocity. For fuller discussion, see M. P. Silverman, (a) *And Yet It Moves: Strange Systems and Subtle Questions in Physics* (Cambridge University Press, New York, 1993), Chapter 3, or (b) *A Universe of Atoms, An Atom in the Universe* (Springer, New York, 2002) Chapter 5.

Solution to the Generator Paradox

Let us now consider the unconventional dynamo. From the specified symmetry, one can describe the magnetic field configuration most conveniently in terms of a cylindrical coordinate system with radial coordinate ρ, azimuthal angle φ, and vertical coordinate z. Ideally, the magnetic field outside the cylinder has only a radial component $B_\rho(\rho)$ that depends on the radial coordinate. For this to be a viable solution, the divergence of the magnetic field must vanish, a statement which, for the given symmetry, takes the mathematical form

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho B_\rho\right) = 0 \ . \tag{5}$$

Eq. (5) tells us that ρB_ρ must be a constant (which I denote by β), and therefore the spatial dependence of the field is

$$B_\rho(\rho) = \frac{\beta}{\rho} \ . \tag{6}$$

Inside the cylinder the spatial variation will be different, since the magnetic field cannot become infinite along the vertical axis ($\rho = 0$), but the symmetry is still cylindrical.

If the conducting loop is fixed (as shown in Figure 1) and the magnet is rotated about the z-axis, there can be no change is magnetic flux through the loop boundary. Indeed, because the field lines are oriented radially, there is no flux through the circuit at all since **B** lies in the plane of the loop—i.e. the scalar product $\mathbf{B}\cdot\mathbf{n}$ vanishes. Hence the emf registered by the galvanometer G is zero.

If the magnet is fixed and the loop ACDJ is rotated about the z-axis, one might suppose from the preceding discussion that the flux through the conducting loop remains zero, since the magnetic field is still radial irrespective of the loop orientation. However, the flux through the loop does change. As segment CD moves in the azimuthal direction (indicated at the instant shown in Figure 1 by the velocity **v**), it opens up a cylindrical surface area (of radius r and length h) to which **B** is everywhere perpendicular. Since $\partial\mathbf{B}/\partial t = 0$, the change in flux through this area is given by the second term in Eq. (3)

$$\mathcal{E} = \oint_{\mathcal{C}} \mathbf{v}\times\mathbf{B}\cdot\mathbf{ds} = \int_{\text{point}\,C}^{\text{point}\,D} v(r)\,B_\rho(r)\,dz = \omega\,\beta\,h \ . \tag{7}$$

To evaluate the contour integral in Eq. (7), I chose **n** (the outward unit normal vector of the rotating part of the surface Σ) to point in the azimuthal direction, in which case the contour $\mathcal{C}$ is to be traversed in a clockwise sense (as judged by someone looking at Figure 1). Thus, the cross product $\mathbf{v}\times\mathbf{B}$ is parallel to the line element **ds** along the segment CD and is perpendicular to **ds** along segments AC and DJ. Moreover, the velocity **v** vanishes along all parts of the fixed contour (containing the galvanometer). Thus, the contour integral in Eq. (7) reduces to the second expression integrated over segment CD only, which, upon substitution of $B_\rho(r) = \beta/r$ [relation (6) evaluated at the location of segment CD], yields $\omega\,\beta\,h$ in which $\omega = d\varphi/dt = v/r$ is the constant angular frequency of rotation of segment CD.

Finally, if the magnet and the loop rotate together, the situation is unchanged from the preceding case. Given the cylindrical symmetry of the magnetic field, it is immaterial whether the magnet rotates or not; the emf that is registered on the galvanometer derives from the rotation of the loop.

The Unconventional Voltage Measurement—Another Induction Puzzle

The unconventional dynamo generates an emf under circumstances that are initially surprising, but which have been accounted for in the preceding section. However, when the phenomenon of induction is involved, the measurement (rather than the generation) of an emf can also be surprising, at least initially. Figure 2 shows a simple circuit comprising a battery with emf $\mathcal{E}_1$ and a resistor with resistance R_1. For simplicity, I will refer to each resistor in this system by the value of its resistance. To measure the potential difference across R_1, one attaches the leads of a voltmeter (a galvanometer with high resistance R_g) at the points a and b. Can it possibly matter whether the meter is placed on the left or the right side of R_1?

Ordinarily, the preceding question would seem ludicrous. If physicists had to worry where on a workbench they placed their voltmeters, electrical science would be in a muddle. However, if a time-varying magnetic flux $\Phi(t)$ (such as produced by a solenoid bearing ac current) threaded through the center of the circuit, as shown in Figure 2, the question has an affirmative answer. Indeed, depending on circumstances, the meter can register zero volts when placed on the right side and a nonzero voltage when placed on the left!

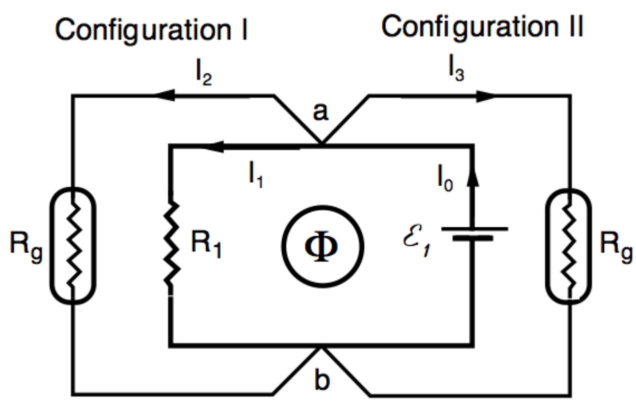

Figure 2: Circuit comprising a resistor R_1 and battery $\mathcal{E}_1$ threaded by a time-varying magnetic flux $\mathbf{\Phi}$. The potential difference across points a and b registered by a galvanometer with internal resistance R_g depends on whether the instrument is placed to the left (Configuration I) or to the right (Configuration II) of the test points.

Consider Configuration I. A current I_0 is drawn from the battery, whereupon a portion I_1 passes through R_1 and a portion I_2 passes through the meter. From Kirchhoff's current law, we have

$$I_0 = I_1 + I_2 \tag{8}$$

at junction a or b. If the unit normal **n** to the plane of the circuit in Figure 2 is directed out of the page, then the positive sense around the circuit is counterclockwise, and the application of Kirchhoff's voltage law to loop a-R_1-b-$\mathcal{E}_1$ leads to

$$\mathcal{E}_1 - I_1 R_1 = \mathcal{E} \tag{9}$$

and to loop a-R_g-b-$\mathcal{E}_1$ leads to

$$\mathcal{E}_1 - I_2 R_g = \mathcal{E} \ , \tag{10}$$

where

$$\mathcal{E} = -\frac{d\Phi}{dt} = \mathcal{E}_0 \sin(\Omega t) \tag{11}$$

is the emf produced in any closed loop through which the flux change occurs. For simplicity, I have assumed a flux oscillating co-sinusoidally at an angular rate Ω radians/second. From Eq. (10), it follows that the meter registers a potential difference

$$V_{ab}^{I} = I_2 R_g = \mathcal{E}_1 - \mathcal{E} \ . \tag{12}$$

Consider next Configuration II. The current I_0', drawn from the battery, divides at junction a, where now a conceivably different portion I_3 passes through the meter. In place of Eq. (8), one then has

$$I_0' = I_1 + I_3 \ . \tag{13}$$

Kirchhoff's voltage law applied to loop a-R_1-b-$\mathcal{E}_1$ is still Eq. (9), but yields

$$I_3 R_g - I_1 R_1 = \mathcal{E} \tag{14}$$

when applied to loop a-R_g-b-R_1. (Note that I_1 and I_3 traverse the loop in opposite senses.) Now, from solutions of Eqs. (9) and (14), one finds that the meter registers the potential

$$V_{ab}^{II} = I_3 R_g = \mathcal{E}_1 \tag{15}$$

independent of the induced emf. This result [Eq. (15)] could have been obtained directly by applying Kirchhoff's law to loop $\mathcal{E}_1$-a-R_g-b through which *no* flux passes.

The two meter positions become equivalent, as expected, only if the induced emf $\mathcal{E}$ vanishes. On the other hand, if one removes the battery from the circuit, in which case $\mathcal{E}_1 = 0$, then the meter reads 0 in Configuration II and $-\mathcal{E}$ in Configuration I. In general, one does not measure instantaneous ac voltage, but the root-mean-square (rms) ac voltage instead. It is not difficult to show that Eqs. (11), (12) and (15) lead to the rms voltages:

$$V_{rms}^{I} = \sqrt{\mathcal{E}_1^2 + \tfrac{1}{2}\mathcal{E}_0^2} \tag{16}$$

$$V_{rms}^{II} = \mathcal{E}_1 \ , \tag{17}$$

where rms voltage is defined by the relation $V_{rms} = \sqrt{\frac{1}{T}\int_0^T V(t)^2 \, dt}$ with $T = 2\pi/\Omega$ the period of oscillation. There is no distinction between instantaneous and rms voltage in Configuration II, because the meter records only the emf of the battery.

Finally, it is to be noted for completeness that the current drawn from the battery also depends on the meter placement. This current is

$$I_0 = \left(\mathcal{E}_1 - \mathcal{E}\right)\left(\frac{1}{R_1} + \frac{1}{R_g}\right) \tag{18}$$

for Configuration I and

$$I_0' = \frac{\mathcal{E}_1 - \mathcal{E}}{R_1} - \frac{\mathcal{E}_1}{R_g} \tag{19}$$

for Configuration II. In the limit that the resistance of the galvanometer becomes infinitely large, the two currents approach the same value $\left(\mathcal{E}_1 - \mathcal{E}\right)/R_1$.

The curious asymmetry in voltage measurement illustrated in this section shows that, when there is a varying magnetic flux through a circuit, it is no longer physically meaningful to assign a unique potential (up to an arbitrary constant) to each point of the circuit. Students who have encountered Faraday's law undoubtedly have heard this warning. But there is often a considerable gap between knowing something and understanding it—hence the reason why long familiar principles of physics can still lead to instructive surprises.