Enjoy the Good Sound of Coke™ —The Art of Modeling by Analogy

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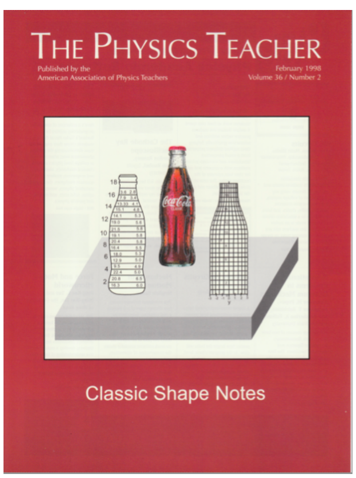
> Whoever wishes to acquire a deep acquaintance with Nature must observe that there are analogies which connect whole branches of science in a parallel manner, and enable us to infer of one class of phenomena what we know of another.
>
> —William Stanley Jevons, *Principles of Science* (1874)

Summary

There is a certain satisfaction that comes from understanding the behaviour of familiar objects. Who has not blown air across a bottle to make a musical sound? As a source of sound, a Coke bottle may resemble a curvy open-ended organ pipe—but that would be a completely erroneous way to think of it. Bizarre as it may seem, I show that a Coke bottle is more accurately modeled as an electrical circuit with inductance and capacitance. Tested experimentally by one of my students, the model closely reproduces the relation beween the resonant sound frequency and level of water in the bottle.



Sounds and Circuits

In the lighthearted, madcap African satire, *The Gods Must Be Crazy,* a Coke bottle, nonchalantly tossed from the cockpit of an airplane, landed in the midst of an isolated Bushman family never before exposed to the familiar commodities of 'civilisation'. Of the many uses the family found for this mysterious 'heaven-sent' gift, among the most pleasing was that of a musical instrument. (As the story unfolded, however, there were other *less* pleasing attributes—and the resourceful Bushman went to great lengths to return the gift and recover his peace of mind.) By teaching courses based on what I have called 'self-directed learning'[1]—the radical proposition that students learn science better when striving to answer questions that arise out of their own curiosity—I have often been led to explore imaginative avenues of physics that would not likely have occurred to me, had it not been for the curiosity of some student. In this way I, together with a student colleague (E. R. Worthy), likewise came to realise that a Coke bottle—or, more precisely, about ten bottles containing different volumes of water—does indeed make a splendid instrument. Yet, surprisingly, for so superficially simple a structure, the

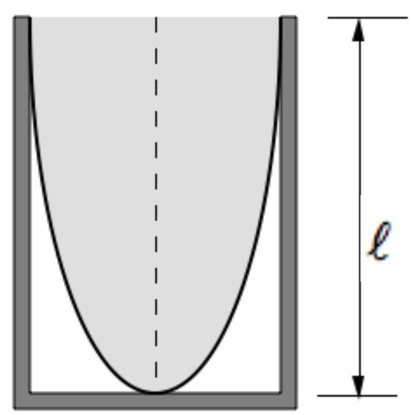


Figure 1: Fundamental mode of a single open-ended organ pipe.

[1] M. P. Silverman, (a) Self-Directed Learning: A Heretical Experiment in Teaching Physics, *American Journal of Physics* **63** (1995) 495; (b) Self-Directed Learning: Philosophy and Implementation, *Science & Education* **5** (1996) 357, (c) Problem-Based Learning and Self-Directed Learning, *What Works II: Postsecondary Education in the 21st Century* (Penn State University, State College PA, 1996).

tones of the bottle are by no means easily accounted for. For my student and me, as for the Bushman, the Coke bottle has not been drained of all its mystery.

To exploit the musical properties of the Coke bottle (or any other bottle) as an acoustic resonator, one must determine the relationship between the fundamental frequency f and the length of the air column ℓ. Despite the overall cylindrical symmetry of the bottle, the problem is a challenging one—and within the elementary physics literature that my student and I surveyed we encountered no discussion of the issues involved beyond the standard geometric depiction of axial standing waves in tubes of constant cross section. As shown in Figure 1 for the case of a tube sealed at one end (like the Coke bottle), each longitudinal mode has a displacement node at the closed end and (to good approximation) a displacement antinode at the open end. The lowest-frequency mode, therefore, has a wavelength $\ell = \frac{1}{4}\lambda$, from which it readily follows that the fundamental frequency is

$$f = \frac{v_s}{4\ell} \tag{1}$$

where v_s is the speed of sound, which is about 344 m/s at a pressure of 1 atm and temperature of 20 °C.

From the shape of a Coke bottle, illustrated in Figure 2a, one might think that the 'organ pipe' of Figure 1 would serve as a useful model for predicting the fundamental frequency. This, however, is far from the case. Nevertheless, a relatively simple approach that avoids solving the differential equations of wave theory can be made by analogy between an acoustic resonator and the ordinarily more familiar elements of ac circuit analysis. A comprehensive wave analysis of acoustic systems leads to equations of the same form as those of ac circuit theory when the lengths of individual components are small compared with a wavelength of sound. Justification of this assertion is by no means trivial, but is demonstrated in advanced books on theoretical acoustics.[2] From such a comparison we find that

(a) the gauge pressure (the difference between actual and equilibrium air pressures) in the acoustic system corresponds to the voltage at a point in the circuit;

(b) the air flow (volume/time) through an orifice corresponds to the electric current;

(c) a short narrow tube (a constriction) of length ℓ_c and cross section S_c is equivalent to an inductance (termed the analogous inductance)

$$L_a = \rho \ell_c / S_c \tag{2}$$

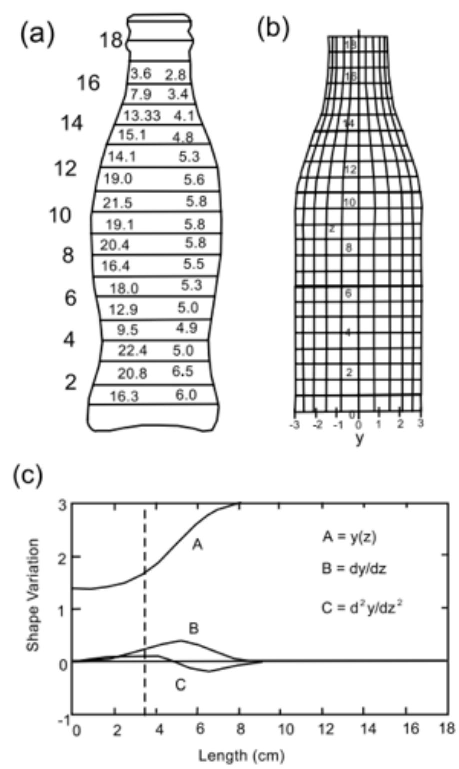


Figure 2: (a) Scale drawing of a Coke bottle; the three sets of numerals designate (left to right) the inside air column length in cm, the water volume of cylindrical segment in cm³, and the bottle diameter in cm. (b) Generation of a Coke bottle from a hyperbolic tangent curve. (c) Variation with length along the bottle of the generatrix and its first and second derivatives.

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[2]L. L. Berenek, *Acoustics* (McGraw-Hill, New York, 1954) 128-143; P. M. Morse, *Vibration and Sound* (American Institute of Physics, New York, 1976) 233 *ff*.

in which ρ is the mass density of air (~1.2 kg/m^3 at 1 atm and 20 °C);

(d) a broad tube (a tank) of length ℓ_t and cross section S_t is equivalent to the analogous capacitance

$$C_a = \frac{S_t \ell_t}{\rho v_s^2} \ , \tag{3}$$

and

(e) radiation of sound (of angular frequency ω) from a constriction opening into free space constitutes an analogous terminal resistance

$$R_a = \frac{\rho \omega^2}{2\pi v_s} \ . \tag{4}$$

One further refinement is necessary to make the model correspond more closely to reality. Because the antinode of a standing wave in a tube actually lies a little above the open end, we should replace ℓ_c in Eq. (2) by an effective length

$$\ell_e = \ell_c + 0.8\sqrt{S_c} \tag{5}$$

that depends on the size of the opening. The correction, which is not necessary for the (much larger) tank, shows that even a flat aperture $(\ell_c = 0)$ contributes an analogous inductance.

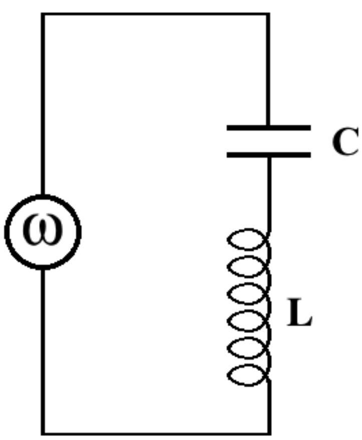

Figure 3: Diagram of the series LC circuit to which the Coke bottle, modeled as a Helmholtz resonator, corresponds.

In circuits that obey Ohm's law, the potential difference V across a circuit element and the current I that flows through the element are linearly related, $V = IZ$, in which the coefficient of proportionality $Z = R + iX$ is called the impedance, comprising a real part (resistance) and an imaginary part (reactance). If the element dissipates energy as heat, then Z is just the familiar real-valued resistance R. However, the element may store energy in an electric field (or equivalently as charge on conducting plates) or in a magnetic field (or equivalently as current through a solenoid), in which case Z is a purely imaginary-valued capacitive or inductive reactance. A real circuit element may exhibit both resistance and reactance to varying degrees depending on the frequency of the electromagnetic signal it is carrying.

With the preceding relations, a wide variety of acoustic systems (bottles, horns, reed instruments, strings, loudspeakers, etc.) can be modeled in terms of their electrical counterparts. Now, let us examine the Coke bottle.

The AC Circuit Model of a Coke Bottle

To an approximation sufficient for the purposes of this discussion, the Coke bottle (8 fluid ounces) of Figure 2a comprises a short neck inserted into a longer tank, a structure known as a Helmholtz resonator. Blowing across the mouth of the bottle excites the air inside to vibrate, but only those vibrations at the resonant frequencies of the bottle are amplified. Unless the bottle is 'overblown', it is principally the fundamental tone that one hears, and it is this tone alone that we want to predict. If we neglect energy dissipation at the open end and at the walls, the bottle can be modeled by the ac circuit of Figure 3 containing a capacitor (of capacitance *C*) and inductor (of inductance *L*) in series. A series LC circuit exhibits a complex impedance $Z = i(X_C - X_L)$ in which

$$X_C = 1/\omega C \tag{6}$$

is the capacitive reactance and

$$X_L = \omega L \tag{7}$$

is the inductive reactance for an applied harmonic signal of angular frequency ω. If the capacitive and inductance reactances are equal, then the impedance of the circuit vanishes. From Eqs. (6) and (7) it follows that this resonance condition occurs at the frequency

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \quad . \tag{8}$$

Substitution of relations (2) and (3) and effective length (5) into Eq. (8) leads to the following expression for the fundamental frequency of a cylindrical bottle

$$f = \frac{v_s}{2\pi}\sqrt{\frac{S_c}{S_t \ell_e \ell_t}} = \frac{v_s\, a}{2\pi\, b}\frac{1}{\sqrt{\left(\ell_c + 0.8a\sqrt{\pi}\right)\ell_t}} \tag{9}$$

with *a* and *b* the radii of the mouth and the base of the bottle, respectively. Since *a*, *b*, and ℓ_c are fixed parameters for a particular bottle, Eq. (9) expresses the fundamental frequency *f* as a function of the variable tank length $\ell_t = \ell - \ell_c$, where ℓ is the full length of the air column. Thus, in marked contrast to Eq. (1) for the constant-diameter organ pipe in which $f \propto \ell^{-1}$, the fundamental of the bottle should vary as $\left(\ell - \ell_c\right)^{-1/2}$.

Now the Coke bottle, whose radius varies smoothly from mouth (*a* ~ 1.4 cm) to base (*b* ~ 3 cm), is not strictly speaking a Helmholtz resonator which, technically, comprises two joined tubes each of constant radius. How, then, is one to decide where the constriction ends and the tank begins? A good rule, supported by examination of the equations characterising wave propagation in the bottle, is as follows: Take ℓ_c to be the distance from the mouth to the point where the second derivative of the bottle shape is maximum. Briefly, the wave equation for sound produced by the bottle differs from the comparable equation for an organ pipe only by a term containing this second derivative. Although small in magnitude and effectively applicable only over a small segment of the bottle length, this term is responsible for the marked difference in acoustic behaviour between the bottle and the organ pipe.

As an illustration, look at Figure 2b, which depicts a mathematical simulation of the Coke bottle obtained by rotating the curve

$$y(z) = 1.6\ \tanh\left(\frac{(z-18)^3}{216}\right) + 1.4 \tag{10}$$

(with radius *y* and length *z* in cm) about the symmetry axis. Apart from the acoustically unimportant shallow indentation near the base in Figure 2a, the generatrix (10) provides an accurate representation of the size and shape of the Coke bottle. Eq. (10) was obtained by trial and error, guided by the ‘principle of simplicity' to select the simplest curve that makes a smooth transition between the mouth and base. The desired sigmoid shape almost cried out for a hyperbolic tangent; the third power of the argument best reproduced the curvature of the bottle in the critical region where constriction joins tank. Figure 2c shows the variation with length of the generatrix (curve A) and its first derivative (curve B) and second derivative (curve C). The location of the positive maximum value of curve C establishes that $\ell_c \sim 3.5$ cm.

Upon substituting into Eq. (9) the preceding Coke-bottle parameters and the speed of sound in air at room temperature, One obtains the final relation

$$f = \frac{1089}{\sqrt{\ell - 3.5}} \text{ Hz} \tag{11}$$

for fundamental frequency f (in Hz) as a function of air column ℓ (in cm).

The Sound of Coke

So, what is the 'sound' of Coke? To test the predictive accuracy of my model, Eqs. (9)-(11), my student measured the frequency of the tones obtained by blowing across the mouth of a Coke bottle filled to different levels of water. In keeping with the spirit of a home-based project to be performed with apparatus more or less readily available outside the physics laboratory, the resonant frequencies of the bottle were measured by means of a guitar tuner calibrated against a well-tuned piano. Water levels were sought for which the tuner registered standard notes, which were then converted to the corresponding frequencies. Heights were measured to within 0.1 cm, and the experimental uncertainty in frequency is estimated from the intervals of the guitar tuner to be less than $(2^{1/48} - 1)$ times the frequency of middle C, or approximately 4 Hz.

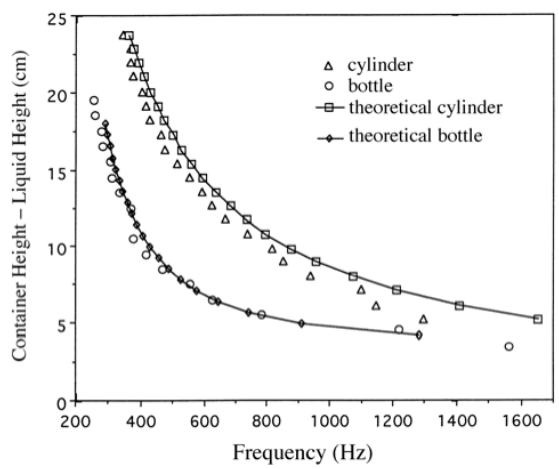

Figure 4: Theoretical and experimental variations of air-column length with frequency for both a Coke bottle and a right-circular cylinder closed at one end. At the scale shown, the uncertainties in length and frequency are smaller than the plotting symbols.

Figure 4, which gives results for both the Coke bottle and a right-circular cylinder closed at one end, summarises the outcome of these experiments. It is readily seen that the observed frequencies of the Coke bottle bear out very well the ac circuit resonator model, and that a model of the bottle as an organ pipe is in thorough disagreement with experiment even though the curvature of the Coke bottle is relatively small (as shown in Figure 2c).

For the reader interested in the musicality of the bottle, the table to the right records the water levels (in cm) required to produce notes needed for various popular tunes. The designation C_4 is 'middle C' on the tempered scale, nominally corresponding to a frequency of 261.6 Hz. Each succeeding half-tone (C#, D, D#, E, etc.) in the octave between C_4 and C_5 is theoretically higher in frequency than the preceding half-tone by the factor

$C_4 = 1.5$ cm		$C_5 = 10.2$ cm
$D_4 = 3.4$		$D_5 = 11.1$
$E_4 = 5.4$		$E_5 = 12.1$
$F_4 = 6.4$		$F_5 = 12.4$
$G_4 = 8.3$		$G_5 = 13.1$

$2^{\frac{1}{12}} = 1.0595$. Music, however, is not a precise science like physics, and a musician will play tones the way they sound best. The water levels that produce the notes listed above correspond only approximately to the frequencies of the tempered scale.

Two familiar tunes played by my student to the class on a set of Coke bottles were

- 'Mary Had a Little Lamb':
    - EDCDEEE DDD EGG EDCDEEE EDDEDC

and

- 'Jingle Bells':
    - EEE EEE EGCD E FFF FEEEE EDDE DG
    - EEE EEE EGCD E FFF FEE GGFD C

Although a general discussion of environmental effects on the tones of the Coke bottle would take us too far afield, the temperature T is sufficiently important to consider briefly here. All other parameters remaining unchanged, the increase in sound velocity v_s with T would raise the pitch of the bottle, as shown explicitly in Eqs. (1) and (9). However, raising T causes both the glass container and water contents to expand, thereby changing the length of the air column. Since the volume coefficient of expansion of water (2.1×10^{-4} per $°\mathrm{C}$) is greater than that of glass ($\sim 1.1\times10^{-5}$ per $°\mathrm{C}$ for Pyrex), an increase in temperature should lead to a shorter air column and therefore to a higher pitch. In our experiments we measured the variation in frequency as a function of temperature and found overall an increase of 22 Hz (approximately the interval of one note) over the range of $85\ °\mathrm{C}$—a change corresponding to a net rise in water level of about 1 cm.

Since a Coke bottle is not of uniform diameter, the change in water level (and therefore the length of the air column) engendered by a given volume expansion will have greater consequence where the bottle is narrow than where it is wide. Thus, temperature variations will affect more severely those pitches in the higher octaves than in the lower ones.

Art of Modeling

Although much of the physics that excites a student's imagination may often pertain to exotic realms far from daily reality (like black holes, time travel, and the fate of the universe), there is also a certain satisfaction that comes from being able to understand the behaviour of familiar objects. Learning physics, I believe, is greatly facilitated when teachers can convey an appreciation for the power of general physical principles to account for what students frequently experience, yet rarely understand.

It may turn out as well—and such is the case with the Coke bottle—that for all its familiarity a superficially simple object hardly worth noting may pose a daunting challenge. In such instances the use of analogy provides a powerful strategy. If science, as Nobelist Peter Medawar has written, is the 'Art of the Soluble'[3], then the *art* of that art is modeling, the capacity to exploit threads of commonality between outwardly dissimilar systems to arrive at a partial understanding of a complex and puzzling phenomenon. Moreover, in this art of modeling what best serves as a model system can be surprising, at least to the uninitiated. Without prior experience very few students—even physics graduate students—would look at

[3]P. B. Medawar, *The Art of the Soluble* (Methuen, London, 1967).

a Coke bottle and see a resonant LC circuit rather than the structurally closer counterpart of an organ pipe

But analogy is not identity—and what a model omits from first consideration may yet prove decisive to deeper enquiry. The study of the humble Coke bottle is by no means a closed book. For example, if the bottle were of soft plastic, then a gentle deformation by squeezing would damp out the fundamental tone. Why? Is this the ineluctable consequence of broken cylindrical symmetry? No, for hard glass bottles of highly elliptical cross section render strong fundamental tones. (Try blowing across an empty maple-syrup bottle.) The simple resonator model does not explain this.

On the other hand, one might ask why the LC circuit model works as well as it does. In a puzzling reversal of expectations, I was initially astonished to discover that supposedly more sophisticated mathematical models of acoustic resonators that treated the bottle as a continuous transmission line with no arbitrary division between neck and tank predicted fundamental tones *less* accurate than those of the cruder resonator model with lumped circuit elements. How can this be? It must suffice here to say only that to understand variable-diameter resonators like a Coke bottle in all their complexity constitutes a study of three-dimensional waves including both radial and longitudinal modes of air vibration as well as the elastic properties of the vessel walls. The investigation is a fascinating one, but not recommended for the mathematically fainthearted.

From the perspective of experiment, however, the ready availability of personal computers with microphones and sound-analysing software makes it possible to explore the acoustic properties of bottles and other simple resonators in great detail. It is an excellent way to learn about waves and vibrations in familiar systems with mysteries yet to be explored.